Given that any loading configuration in a cracked body can be represented by equivalent crack face tractions, the general mixed-mode 3D formulation of the weight function approach can be expressed in the following form:

$$K_a(\eta) = \int_{S_c} T_i h_a(x_i, \eta)\, dS \tag{2.61}$$

where $T_i$ are the tractions assumed to act on the crack surface, $S_c$.

See Chapter 9 for examples of practical applications of weight functions.

## 2.7 Relationship between *K* and $\mathcal{G}$

Two parameters that describe the behavior of cracks have been introduced so far: the energy release rate and the stress intensity factor. The former parameter quantifies the net change in potential energy that accompanies an increment of crack extension; the latter quantity characterizes the stresses, strains, and displacements near the crack tip. The energy release rate describes the global behavior, while $K$ is a local parameter. For linear elastic materials, $K$ and $\mathcal{G}$ are uniquely related.

For a through crack in an infinite plate subject to a uniform tensile stress (Figure 2.3), $\mathcal{G}$ and $K_I$ are given by Equations 2.27 and 2.48, respectively. Combining these two equations leads to the following relationship between $\mathcal{G}$ and $K_I$ for plane stress:

$$\mathcal{G} = \frac{K_I^2}{E} \tag{2.62}$$

For plane strain conditions, $E$ must be replaced by $E/(1 - \nu^2)$. To avoid writing separate expressions for plane stress and plane strain, the following notation will be adopted throughout this book:

$$E' = E \quad \text{for plane stress} \tag{2.63}$$

and

$$E' = \frac{E}{1 - \nu^2} \quad \text{for plane strain} \tag{2.64}$$

Thus the $\mathcal{G} - K_I$ relationship for both plane stress and plane strain becomes

$$\mathcal{G} = \frac{K_I^2}{E'} \tag{2.65}$$

Since Equations 2.27 and 2.48 apply only to a through crack in an infinite plate, we have yet to prove that Equation 2.65 is a general relationship that applies to all configurations.

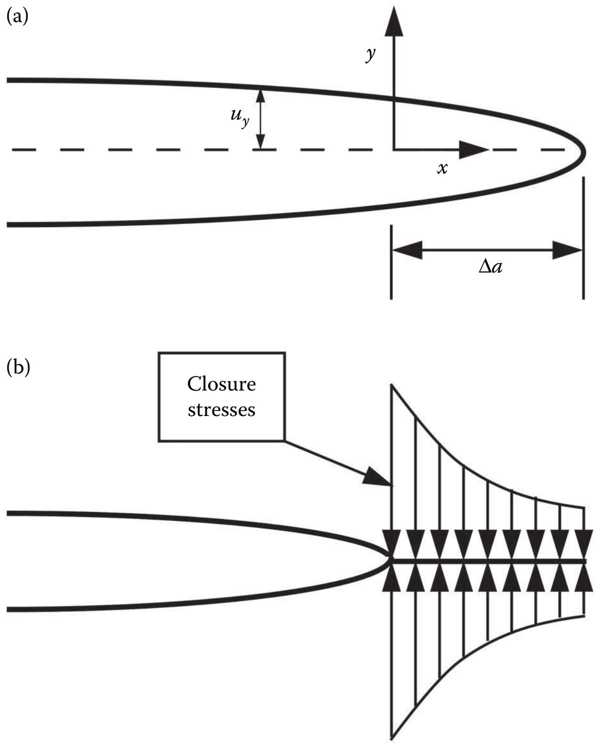


**FIGURE 2.28**
Application of closure stresses which shorten a crack by $\Delta a$.

Irwin [9] performed a crack closure analysis that provides such a proof, which is presented below.

Consider a crack of initial length $a + \Delta a$ subject to Mode I loading, as illustrated in Figure 2.28a. It is convenient in this case to place the origin a distance $\Delta a$ behind the crack tip. Assume that the plate has unit thickness. Let us now apply a compressive stress field to the crack faces between $x = 0$ and $\Delta a$ of sufficient magnitude to close the crack in this region, as Figure 2.28b illustrates work required to close the crack at the tip is related to the energy release rate:

$$\mathcal{G} = \lim_{\Delta a \to 0} \left( \frac{\Delta U}{\Delta a} \right)_{\text{fixed load}} \tag{2.66}$$

Here $\Delta U$ is the work of crack closure, which is equal to the sum of contributions to work from $x = 0$ to $\Delta a$:

$$\Delta U = \int_{x=0}^{x=\Delta a} dU(x) \tag{2.67}$$

and the incremental work at $x$ is equal to the area under the force–displacement curve:

$$dU(x) = 2\frac{1}{2} F_y(x) u_y(x) = \sigma_{yy}(x) u_y(x)\, dx \tag{2.68}$$

The factor of 2 on work is required because both crack faces are displaced an absolute distance $u_y(x)$. The crack opening displacement, $u_y$, for Mode I is obtained from Table 2.2 by setting $\theta = \pi$:

$$u_y = \frac{(\kappa+1)K_I(a+\Delta a)}{2\mu}\sqrt{\frac{\Delta a - x}{2\pi}} \tag{2.69}$$

Here $K_I(a + \Delta a)$ denotes the stress intensity factor at the original crack tip. The normal stress required to close the crack is related to $K_I$ for the shortened crack:

$$\sigma_{yy} = \frac{K_I(a)}{\sqrt{2\pi x}} \tag{2.70}$$

Combining Equations 2.66 through 2.70 gives

$$\begin{aligned} \mathcal{G} &= \lim_{\Delta a \to 0} \frac{(\kappa+1)K_I(a)K_I(a+\Delta a)}{4\pi\mu\Delta a}\int_0^{\Delta a}\sqrt{\frac{\Delta a - x}{x}}\,dx \\ &= \frac{(\kappa+1)K_I^2}{8\mu} = \frac{K_I^2}{E'} \end{aligned} \tag{2.71}$$

Thus, Equation 2.65 is a general relationship for Mode I. The above analysis can be repeated for other modes of loading; the relevant closure stress and displacement for Mode II is $\tau_{yx}$ and $u_x$ and the corresponding quantities for Mode III are $\tau_{yz}$ and $u_z$. When all three modes of loading are present, the energy release rate is given by

$$\mathcal{G} = \frac{K_I^2}{E'} + \frac{K_{II}^2}{E'} + \frac{K_{III}^2}{2\mu} \tag{2.72}$$

Contributions to $\mathcal{G}$ from the three modes are additive because energy release rate, like energy, is a scalar quantity. Equation 2.72, however, assumes self-similar crack growth; that is, a planar crack is assumed to remain planar and maintain a constant shape as it grows. Such is usually not the case for mixed-mode fracture. See Section 2.11 for further discussion of energy release rate in mixed-mode problems.

## 2.8 Crack Tip Plasticity

Linear elastic stress analysis of sharp cracks predicts infinite stresses at the crack tip. In real materials, however, stresses at the crack tip are finite because the crack tip radius must be finite (Section 2.2). Inelastic material deformation, such as plasticity in metals and crazing in polymers, leads to further relaxation of crack tip stresses.

The elastic stress analysis becomes increasingly inaccurate as the inelastic region at the crack tip grows. Simple corrections to LEFM are available when moderate crack tip